Correspondence between nonrelativistic anti-de Sitter space and conformal field theory, and aging-gravity duality

Djordje Minic* and Michel Pleimling†
Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0435, USA
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We point out that the recent discussion of nonrelativistic anti-de Sitter space and conformal field theory correspondence has a direct application in nonequilibrium statistical physics, a fact which has not been emphasized in the recent literature on the subject. In particular, we propose a duality between aging in systems far from equilibrium characterized by the dynamical exponent $z=2$ and gravity in a specific background. The key ingredient in our proposal is the recent geometric realization of the Schrödinger group. We also discuss the relevance of the proposed correspondence for the more general aging phenomena in systems where the value of the dynamical exponent is different from 2.

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I. INTRODUCTION

The problem of understanding the behavior of systems far from equilibrium is one of the most challenging questions in contemporary many-body physics [1]. In this context it turns out that local scale invariance [2,3] can play a very important role in understanding dynamical scaling aspects of systems far from equilibrium [1,4]. In particular, in the discussions of the phenomenon of aging [5], in other words the observation that properties of nonequilibrium systems generically depend on the time since the system was brought out of equilibrium, the role of the nonrelativistic conformal group, i.e., Schrödinger group, has been found to be crucial in systems with a dynamical exponent $z=2$ [1–3,6,7]. In this paper we propose that aging phenomena with the dynamical exponent $z=2$ can be understood from a dual, gravitational point of view.

In this context it is useful to recall that recently there has been a lot of activity regarding the possible relevance of the anti-de Sitter space and conformal field theory (AdS/CFT) correspondence to other difficult condensed matter problems such as high temperature superconductivity [8]. The general concept of duality, on which a lot of recent progress in quantum field theory and string theory is based, has also been used recently in various condensed matter settings [9]. Most recently Son [10] and independently Balasubramanian and McGreevy [11] have proposed a geometrical realization of the Schrödinger group in what has been called [10] "an AdS/cold atoms correspondence."

In this paper we wish to apply this geometric realization of the Schrödinger group to an aging-gravity correspondence. We consider this as a first step in an approach to understanding the nonequilibrium dynamics of more general systems with $z\neq 2$, and especially of disordered systems [1,4,12]. On one side we aim to bring the topic of aging in systems far from equilibrium to the attention of string and field theorists interested in applications of nonrelativistic AdS/CFT duality. Thus Sec. II briefly summarizes the importance of the Schrödinger (nonrelativistic conformal) group in aging in a self-consistent manner. On the other side, we aim to bring the tools of nonrelativistic AdS/CFT to the attention of the physicists working in nonequilibrium statistical physics, and in particular, in aging phenomena. Sections III and IV are devoted to them. By putting together the information from these two areas of physics we propose a duality between aging phenomena and gravitational physics in specific backgrounds. The ultimate aim of this duality is to be able to characterize different universality classes by computing the critical indexes of relevant correlation functions as explained in Sec. II, by using the classical physics of certain fields propagating in gravitationally nontrivial backgrounds as explained in Secs. III and IV. Ultimately, the full nonperturbative correlation functions should be captured by the string theory [13] sigma model in these backgrounds, thus bringing the methods of string theory into nonequilibrium statistical physics and vice versa.

II. AGING PHENOMENA AND THE SCHRÖDINGER GROUP

Let us start by briefly reviewing what is known about the dynamical scaling and scale invariance in dynamical systems with aging [1,14]. The general setup for the study of aging behavior is as follows: One considers a coarse-grained order parameter $\phi(t,\vec{r})$, conjugate to a generalized field $h(t,\vec{r})$, which is usually assumed to be fully disordered at $t=0$, i.e., $\langle\phi(0,\vec{r})\rangle=0$. For a magnetic system, the order parameter is of course the magnetization whereas the conjugate field is an external magnetic field. In the following we consider the case where the order parameter is not conserved by the dynamics (model *A* dynamics). Typically, one studies the scaling behavior of two-time correlation functions (here and in the following we assume that spatial translation invariance holds)

$$C(t,s) = \langle\phi(t,\vec{r})\phi(s,\vec{r})\rangle \sim s^{-b} f_C(t/s) \qquad (1)$$

as well as of two-time response functions

*dminic@vt.edu
†michel.pleimling@vt.edu

$$R(t,s) = \frac{\delta\langle\phi(t,\vec{r})\rangle}{\delta h(s,\vec{r})} \sim s^{-1-a} f_R(t/s), \quad (2)$$

where a and b are nonequilibrium exponents whereas f_C and f_R are scaling functions. This scaling behavior is expected in the aging regime, defined by both t and s as well as $t-s$ being much larger than the characteristic microscopic time scale. Note also that this scaling assumes a single characteristic length scale L which scales with time t as

$$L(t) \sim t^{1/z}, \quad (3)$$

where z is the dynamical exponent. For large values of the argument one expects

$$f_C(x) \sim x^{-\lambda_C/z}, \quad f_R(x) \sim x^{-\lambda_R/z} \quad (4)$$

with nonequilibrium exponents λ_C and λ_R. The aging behavior just summarized is called simple or full aging and has been observed in many exactly solvable models, in numerical simulations of more complex models, as well as in actual experiments.

Given the success of conformal invariance in equilibrium critical phenomena, it is natural to ask whether scaling functions and the values of nonequilibrium exponents might be deduced from symmetry principles by invoking generalized dynamical scaling with a space-time dependent scale factor $b=b(t,\vec{r})$. This program has been reviewed in [1] and here we concentrate on the specific case when $L(t)\sim t^{1/2}$, i.e., $z=2$. This is a very important case as it encompasses systems undergoing phase-ordering with nonconserved dynamics [15]. The generalization for $z\neq 2$ has been discussed in [1] and we will briefly comment on this in the concluding remarks of this paper.

The theoretical description of aging systems with a dynamical exponent $z=2$ starts from a stochastic Langevin equation which for a nonconserved order parameter reads

$$2M\partial_t\phi = \nabla^2\phi - \frac{\delta V[\phi]}{\delta\phi} + \eta, \quad (5)$$

where V is a Ginzburg-Landau potential and η is a Gaussian white noise that arises due to the contact with a heat bath [16]. The theory of local scale invariance then permits one to show [1,17] that under rather general conditions all averages (i.e., correlation and response functions) of the noisy theory (5) can be reduced *exactly* to averages of the corresponding deterministic, noiseless theory.

For the $z=2$ case the relevant symmetry structure is the Schrödinger group [18]. It is well known that the free diffusion equation

$$2M\partial_t\phi = \nabla^2\phi \quad (6)$$

is invariant under the Schrödinger group (the free diffusion equation being essentially equivalent to the free Schrödinger equation). The Schrödinger group is defined through the space-time transformations

$$t \to t' = \frac{\alpha t+\beta}{\gamma t+\delta}, \quad \vec{r}\to\vec{r}\,' = \frac{\mathbf{R}\vec{r}+\vec{v}t+\vec{a}}{\gamma t+\delta}, \quad (7)$$

where α, β, γ, δ and $\vec{v}$, $\vec{a}$ are real parameters and $\alpha\delta-\beta\gamma=1$, whereas $\mathbf{R}$ denotes a rotation matrix in d spatial dimensions. The algebra of generators of the Schrödinger group consists of temporal translations H, spatial translations P^I, Galilean transformations $\mathcal{K}^I$, rotations $\mathcal{M}^{IJ}$, dilatations D, and the special conformal transformation C. The algebra of generators of the Schrödinger group [1,10] reads as follows:

$$[\mathcal{M}^{IJ},\mathcal{M}^{KL}] = i(\delta^{IK}\mathcal{M}^{JL}+\delta^{JL}\mathcal{M}^{IK}-\delta^{IL}\mathcal{M}^{JK}-\delta^{JK}\mathcal{M}^{IL}), \quad (8)$$

$$[\mathcal{M}^{IJ},P(\mathcal{K})^K] = i(\delta^{IK}P(\mathcal{K})^J - \delta^{JK}P(\mathcal{K})^I), \quad (9)$$

where $P(\mathcal{K})$ denotes either the momentum P or the Galilean boost $\mathcal{K}$ generator, and

$$[D,P^I] = -iP^I, \quad [D,\mathcal{K}^I] = i\mathcal{K}^I, \quad [P^I,\mathcal{K}^J] = -i\mathcal{M}^{IJ}, \quad (10)$$

and finally

$$[D,H] = -2iH, \quad [C,H] = -iD, \quad [D,C] = 2iC, \quad (11)$$

where D simply rescales t and $\vec{r}$ as $t\to e^{2\rho}t$ and $\vec{r}\to e^{\rho}\vec{r}$ and C acts as $t\to t/(1+\rho t)$ and $\vec{r}\to\vec{r}/(1+\rho t)$.

In complete analogy with the conformal field theory bootstrap [19] one of the immediate consequences of Schrödinger invariance is a restricted form of the two- and three-point functions for the ϕ fields [1]. The two-point function is essentially given by the heat kernel (i.e., Green's function) of the diffusion equation (up to a normalization constant)

$$\langle\phi_1(t_1,\vec{r}_1)\phi_2(t_2,\vec{r}_2)\rangle = \delta_{x_1,x_2}\delta_{M_1+M_2,0}(t_{1,2})^{-x_1}\exp\left(-\frac{M_1}{2}\frac{\vec{r}_{1,2}^{\,2}}{t_{1,2}}\right), \quad (12)$$

where $t_{1,2}\equiv t_1-t_2$ and $\vec{r}_{12}\equiv\vec{r}_1-\vec{r}_2$, with the scaling dimensions x_1 and x_2 and the masses M_1 and M_2. Similarly the three-point function is

$$\langle\phi_1(t_1,\vec{r}_1)\phi_2(t_2,\vec{r}_2)\phi_3(t_3,\vec{r}_3)\rangle = \delta_{M_1+M_2+M_3,0}(t_{1,2})^{-x_{12,3}/2}(t_{2,3})^{-x_{23,1}/2}(t_{1,3})^{-x_{13,2}/2}K, \quad (13)$$

where $t_{i,j}\equiv t_i-t_j$, $\vec{r}_{ij}\equiv\vec{r}_i-\vec{r}_j$, $x_{ij,k}\equiv x_i+x_j-x_k$, and where K is given by

$$K \equiv \exp\left(-\frac{M_1}{2}\frac{\vec{r}_{1,3}^{\,2}}{t_{1,3}}\right)\exp\left(-\frac{M_2}{2}\frac{\vec{r}_{2,3}^{\,2}}{t_{2,3}}\right)F\left(\frac{(\vec{r}_{1,3}t_{2,3}-\vec{r}_{2,3}t_{1,3})^2}{t_{1,2}t_{2,3}t_{1,3}}\right). \quad (14)$$

Here F denotes an arbitrary differentiable function.

These relations have an immediate application to aging: the response function is constrained and the exponents follow. The correlation function is likewise constrained. In deriving expressions for response and correlation functions one has to note that in the aging regime time-translation invariance is broken and that only a subgroup of the Schrödinger group that does not contain time translations has to be taken into account [1,17]. It follows that a quasiprimary scaling operator that transforms covariantly under the aging group is characterized by the triplet (x,ξ,M), where ξ is a "quantum number" associated with the field ϕ (see [1,17,20] for details). In the field-theoretical setting the autoresponse function $R(t,s)$ is written as $R(t,s)=\langle\phi(t)\phi'(s)\rangle$, where the order parameter ϕ and the associated response field ϕ' are characterized by the exponents (x,ξ) and (x',ξ'). From this one obtains the following expression for the autoresponse function [1]:

$$R(t,s)=s^{-(x+x')/2}\left(\frac{t}{s}\right)^{\xi}\left(\frac{t}{s}-1\right)^{-x-2\xi}\Theta(t-s)\,\delta_{x+2\xi,x'+2\xi'}, \tag{15}$$

where $\Theta(x)$ is the Heaviside step function. Comparing with the expected scaling behavior Eqs. (2) and (4) yield for the critical exponents (for $z=2$)

$$\lambda_R=2(x+\xi),\quad 1+a=(x+x')/2. \tag{16}$$

Similarly, for the spatiotemporal response function one obtains the expression

$$R(t,s;\vec{r})=R(t,s)\exp\left(-\frac{M}{2}\frac{\vec{r}^2}{t-s}\right). \tag{17}$$

Finally, explicit expressions can also be derived for the autocorrelation function [7,21]. These expressions are rather cumbersome and will not be reproduced here.

It is important to note that all of these predictions have been tested to yield the exact results in a large variety of exactly solvable models and to describe faithfully over many time decades numerical data obtained for more complex systems [1].

III. SCHRÖDINGER GROUP AND ITS ASSOCIATED GEOMETRY

In this section we discuss an embedding of the Schrödinger group in the conformal group and the natural geometric realization of the Schrödinger group as recently discussed by Son [10]. We here follow the presentation by Son [10] and Balasubramanian and McGreevy [11], whereas an alternative derivation is discussed in [1,22].

Son [10] and Balasubramanian and McGreevy [11] start from a manifestly conformally invariant massless Klein-Gordon equation in $(d+1)+1$- dimensional Minkowski space time,

$$-\partial_t^2\phi+\partial_i^2\phi=0, \tag{18}$$

where the summation of the repeated indices is assumed. Using the light-cone coordinates

$$x^{\pm}\equiv\frac{1}{\sqrt{2}}(x^0\pm x^{d+1}) \tag{19}$$

(a similar definition also holds for other quantities used below), one can rewrite the massless Klein-Gordon equation as

$$(-2\partial_-\partial_++\partial_i^2)\phi=0. \tag{20}$$

By identifying $\partial_-\equiv iM$ the Klein-Gordon equation becomes the Schrödinger equation with x^+ playing the role of time,

$$i\frac{\partial\phi}{\partial x^+}=-\frac{1}{2M}\partial_i^2\phi. \tag{21}$$

The algebraic embedding of the generators of the Schrödinger group into the conformal group follows the embedding of the Schrödinger equation into the massless Klein-Gordon equation [1,10,11,22]. The conformal algebra is given by

$$[m^{IJ},m^{KL}]=i(\eta^{IK}m^{JL}+\eta^{JL}m^{IK}-\eta^{IL}m^{JK}-\eta^{JK}m^{IL}), \tag{22}$$

$$[m^{IJ},p^K]=i(\eta^{IK}p^J-\eta^{JK}p^I),$$

where m^{IJ} denote the rotation and p^I the spatial translation generators as well as

$$[\tilde{d},p^I]=-ip^I,\quad[\tilde{d},k^I]=ik^I,\quad[p^I,k^J]=-2i(\eta^{IJ}\tilde{d}+m^{IJ}), \tag{23}$$

with dilatations $\tilde{d}$ and boosts k^I and $I,J,K,L=0,\ldots,d+1$. One now identifies the light-cone momentum $p^+=-i\partial_-$ with the nonrelativistic mass M as above,

$$p^+\equiv\frac{1}{\sqrt{2}}(p^0+p^{d+1})=M. \tag{24}$$

The operators that commute with p^+ then close into the Schrödinger group

$$H=p^-,\quad P^I=p^I,\quad \mathcal{M}^{IJ}=m^{IJ},\quad \mathcal{K}^I=m^{I+}, \tag{25}$$

$$D=\tilde{d}+m^{+-},\quad 2C=k^+.$$

Based on this algebraic embedding of the Schrödinger group into the conformal group and using the known geometric realization of the conformal group in $d+1$ dimensions in terms of the isometries of the anti-de-Sitter $\mathrm{AdS}_{(d+1)+1}$ space

$$ds^2=\frac{1}{u^2}(du^2+\eta_{IJ}dx^Idx^J), \tag{26}$$

Son [10] and Balasubramanian and McGreevy [11] have recently proposed the following metric which is invariant under $D=\tilde{d}+m^{+-}$ but not under the separate actions of $\tilde{d}$ or m^{+-} as the natural geometric realization of the Schrödinger group [23]:

$$ds^2=-2\frac{(dx^+)^2}{u^{2z}}+\frac{-2dx^+dx^-+dx^idx^i+du^2}{u^2}. \tag{27}$$

Note that this metric is not restricted to the case $z=2$ (the value of the dynamical exponent encountered in pure systems undergoing phase ordering), but encompasses also cases that are Galilean invariant and for which $z\neq 2$.

IV. AGING-GRAVITY DUALITY

In this section we put the two previous sections together to propose an aging-gravity duality. By the aging-gravity duality we mean a precise mathematical correspondence between aging phenomena and gravitational physics in specific backgrounds which capture the geometry of the nonrelativistic conformal group and its subgroups, such as the aging group in its simplest version. The ultimate aim of this correspondence, as already emphasized in the Introduction, is to be able to compute the critical indexes of relevant correlation functions by using the classical physics of certain fields propagating in the gravitationally nontrivial background from Sec. III. Ultimately, the full nonperturbative correlation functions as well as characterization of different universality classes should be captured by the string theory in this background.

The dictionary we propose states that the generating functional of one particle irreducible (1PI) correlation functions of certain operators O relevant for the physics of aging phenomena [in the nonrelativistic CFT (NRCFT)] is equal to the exponent of the action for certain fields φ propagating in a geometric background, evaluated for the boundary values of these fields (φ_b) equal to the sources J for the operators O,

$$Z_{\mathrm{NRCFT}}(J) = e^{-S(\varphi)}, \quad \varphi_b = J. \tag{28}$$

For example, the relevant action for a scalar field φ in the background discussed in Sec. III is [10,11]

$$S = \frac{1}{2}\int d^{d+3}x\sqrt{g}(\partial_I\varphi\partial_J\varphi g^{IJ} - m^2\varphi^2), \tag{29}$$

where $d^{d+3}x \equiv d^d\vec{r}dtdudx^-$ (x^+ being the t coordinate). The equation of motion for φ is

$$\frac{1}{\sqrt{g}}\partial_I(\sqrt{g}g^{IJ}\partial_J\varphi) + m^2\varphi = 0. \tag{30}$$

The solution ansatz is dictated by symmetries

$$\varphi = f(u)e^{i\omega t + i\vec{k}\cdot\vec{r} + iMx^-}, \tag{31}$$

and the radial differential equation for $f(u)$ is [10,11]

$$\left[-r^{d+3}\frac{\partial}{\partial u}\left(r^{-d-1}\frac{\partial}{\partial u}\right) + (2l\omega + \vec{k}^2)r^2 + l^2r^{4-2z} + m^2\right]f(u) = 0, \tag{32}$$

where close to the boundary $f(u) \sim u^{\Delta_\pm}$ with [10,11]

$$\Delta_\pm = 1 + \frac{d}{2} \pm \sqrt{(1+d/2)^2 + m^2 + \delta_{z,2}l^2}. \tag{33}$$

Again, we give here the expression valid for general values of z. Note that for $z=2$ and in $d=3$, which is relevant for the problem of aging in systems undergoing phase-ordering [25],

$$f(u) \sim u^{5/2}K_\nu(ku), \tag{34}$$

where K_ν is the modified Bessel function and

$$\nu = \sqrt{(5/2)^2 + M^2 + m^2}, \quad k^2 \equiv 2M\omega + \vec{k}^2. \tag{35}$$

By using the usual AdS/CFT dictionary [26] one evaluates the on-shell action [10,11] to obtain (after introducing a cutoff near the boundary parametrized by $x_b \equiv \vec{r}, t, x^-$ at $u=\epsilon$)

$$S_0 = \frac{1}{2}\int d^{d+2}x_b\varphi(x_b)\partial_u\varphi(x_b) \tag{36}$$

which in momentum space gives

$$\frac{1}{2}\int dp\varphi(-p)C(k,\epsilon)\varphi(p), \tag{37}$$

where

$$C(k,\epsilon) = \sqrt{g}g^{uu}f(r)\partial_u f(u)|_{u\to\epsilon} \tag{38}$$

with $f(u) \sim K_\nu(ku)$ so that the two point function of our order parameter ϕ (whose source in the functional integral is represented by φ) is essentially given by $C(k,\epsilon)$:

$$\langle\phi_1(\omega_1\vec{k}_1)\phi_2(\omega_2\vec{k}_2)\rangle = \delta(\vec{k}_1 + \vec{k}_2)\epsilon^{-5}[(k^2\epsilon^2)/4]^\nu \tag{39}$$

or in the position space

$$\langle\phi_1(t,\vec{r})\phi_2(0,0)\rangle = \delta_{\Delta_1,\Delta_2}(t)^{-\Delta_1}\exp\left(-\frac{M}{2}\frac{\vec{r}^2}{t}\right), \tag{40}$$

which is precisely what we have found in Sec. II based on the requirements of the Schrödinger invariance.

Thus the geometric interpretation and an AdS/CFT-like dictionary obviously capture the main symmetry constraints and thus also the result for the three-point function follows. Consequently, the predictions for the scaling behavior in aging with $z=2$ follow as well.

V. FUTURE DIRECTIONS

In this paper we have discussed a dictionary between aging phenomena and gravity based on the geometric realization of the Schrödinger group. This discussion was based on the recent discussion of nonrelativistic AdS/CFT duality [10,11,27]. Note that this dictionary is natural from the point of view of the proposed closed relation between quantum gravity and nonequilibrium statistical physics [28]. This aging-gravity dictionary in some sense is an example that extrapolates the Wilsonian dictionary between quantum field theory and equilibrium statistical physics, to quantum gravity (i.e., string theory) in certain backgrounds, and certain nonequilibrium statistical mechanics phenomena.

What is a possible benefit that this geometric approach might have in the future regarding the more detailed understanding of aging in systems far from equilibrium? We note that the example of phase-ordering kinetics encountered in the standard Ising model, which is characterized by $z=2$ [15], is already very interesting. Nevertheless, the geometric approach does offer a possibility for treating systems whose dynamical exponent z is different from 2. It is apparent from Eqs. (27) and (32) that the geometric background dual to the Schrödinger group can be discussed for general z. Of course, the physics of systems with $z=2$ and $z\neq 2$ shows some differences [1,4,12], yet the geometric picture does offer a different point of view on trying to understand the correlation functions in the $z \neq 2$ case. Thus this geometrical picture might prove very useful in order to incorporate into a unifying theoretical framework the recently extensively discussed superuniversality of space-time quantities in disordered ferromagnets [12,29].

Furthermore, other order parameters can be considered even in the $z=2$ case. One could obviously turn on the vector and tensor modes in the same background and examine their correlators. Also, the higher order correlation functions, which can be studied numerically, are amenable to the same geometric treatment. Perhaps even a classification of different dynamical behaviors is possible in this case. We plan to explore these issues in the future.

What is perhaps most interesting is that the geometric realization of the Schrödinger group can be extended to the aging group [1] by considering flows away from the nonrelativistic conformal fixed point. This is a familiar strategy in the domain of the AdS/CFT correspondence. Note that the aging group in the simplest version is a subgroup of the Schrödinger group in which one gets rid of time-translational invariance, a necessary requirement in order to describe nonequilibrium systems out of stationarity.

Finally, one should remember that ultimately one should be dealing with a string theory description in the backgrounds relevant for the Schrödinger or the aging group. In that case, as in approaches to understand other strongly correlated systems such as gauge theories, one ultimately has to deal with the dynamics of nontrivial two-dimensional sigma models in nontrivial backgrounds. Nevertheless, this geometric viewpoint does open a door in the field of aging phenomena in systems far from equilibrium with many exciting and unforeseen applications.

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[1] For a short review, see M. Henkel and M. Pleimling, in *Rugged Free Energy Landscapes: Common Computational Approaches in Spin Glasses, Structural Glasses and Biological Macromolecules*, edited by W. Janke, Lecture Notes in Physics Vol. 736 (Springer, Berlin, 2008), p. 107; e-print arXiv:cond-mat/0703466; for a more exhaustive review, see M. Henkel and M. Pleimling, *Non-Equilibrium Phase Transitions*, Part II (to be published), and references therein.
[2] M. Henkel, J. Stat. Phys. **75**, 1023 (1994).
[3] M. Henkel, Nucl. Phys. B **641**, 405 (2002).
[4] M. Henkel and M. Pleimling, Europhys. Lett. **76**, 561 (2006).
[5] *Ageing and the Glass Transition*, edited by M. Henkel, M. Pleimling, and R. Sanctuary, Lecture Notes in Physics Vol. 716 (Springer, Berlin, 2007).
[6] M. Henkel and M. Pleimling, Phys. Rev. E **68**, 065101 (2003).
[7] M. Henkel, A. Picone, and M. Pleimling, Europhys. Lett. **68**, 191 (2004).
[8] S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007); S. A. Hartnoll and C. P. Herzog, Phys. Rev. D **76**, 106012 (2007); S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. **101**, 031601 (2008); D. Minic and J. J. Heremans, Phys. Rev. B **78**, 214501 (2008); S. S. Gubser and S. S. Pufu, J. High Energy Phys. 11, (2008) 033.
[9] For example, C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son, Phys. Rev. D **75**, 085020 (2007); J. J. Heremans and D. Minic, e-print arXiv:0802.4117; V. Jejjala, D. Minic, Y. J. Ng, and C. H. Tze, Class. Quantum Grav. **25**, 225012 (2008).
[10] D. T. Son, Phys. Rev. D **78**, 046003 (2008).
[11] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101**, 061601 (2008), and references therein.
[12] M. Henkel and M. Pleimling, e-print arXiv:0807.1485, Phys. Rev. B (to be published).
[13] A. Adams, K. Balasubramanian, and J. McGreevy, e-print arXiv:0807.1111; J. Maldacena, D. Martelli, and Y. Tachikawa, J. High Energy Phys. 10, (2008) 072; C. P. Herzog, M. Rangamani, and S. F. Ross, e-print arXiv:0807.1099.
[14] L. F. Cugliandolo, in *Slow Relaxation and Non Equilibrium Dynamics in Condensed Matter*, edited by J.-L. Barrat, J. Dalibard, J. Kurchan, and M. V. Feigel'man (Springer, New York, 2003).
[15] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[16] In some applications, notably in the context of phase-ordering kinetics, it is useful to consider the following extension [17]:

$$2M\partial_t\phi = \nabla^2\phi - \frac{\delta V[\phi]}{\delta\phi} - v(t)\phi + \eta,$$

where $v(t)$ is a time-dependent potential. Obviously, this equation can be reduced to Eq. (5) through a gauge transformation.
[17] A. Picone and M. Henkel, Nucl. Phys. B **688**, 217 (2004).
[18] H. A. Kastrup, Nucl. Phys. B **7**, 545 (1968); C. R. Hagen, Phys. Rev. D **5**, 377 (1972); U. Niederer, Helv. Phys. Acta **47**, 167 (1974).
[19] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
[20] M. Henkel, T. Enss, and M. Pleimling, J. Phys. A **39**, L589 (2006).
[21] M. Henkel and F. Baumann, J. Stat. Mech.: Theory Exp. (2007) P07015.
[22] M. Henkel and J. Unterberger, Nucl. Phys. B **660**, 407 (2003); **746**, 155 (2006).
[23] This metric has been thoroughly studied in a different context in [24].
[24] C. Duval, G. Gibbons, and P. Horváthy, Phys. Rev. D **43**, 3907 (1991), and references therein.
[25] The $d=2$ situation has been also treated in numerical studies. In that case obviously the geometric dual gives $f(u) \sim u^2 K_{\sqrt{4+M^2+m^2}}(u\sqrt{2M\omega+\vec{k}^2})$.

[26] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[27] See, for example, W. D. Goldberger, e-print arXiv:0806.2867; J. L. F. Barbon and C. A. Fuertes, J. High Energy Phys. 09, (2008) 030, and references therein.
[28] V. Jejjala, M. Kavic, and D. Minic, Int. J. Mod. Phys. A **22**, 3317 (2007); V. Jejjala, M. Kavic, D. Minic, and C. H. Tze, e-print arXiv:0804.3598.
[29] A. Sicilia, J. J. Arenzon, A. J. Bray, and L. F. Cugliandolo, Europhys. Lett. **82**, 10001 (2008).